REDISTRIBUTION OF SALTS IN SOIL

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Irrigation of saline soils is a very important method of soil improvement and will continue to be important in the immediate future. Many researchers ([1] and others) have pointed out the importance of predicting the quantitative redistribution of salts and of further theoretical investigation of this problem. Hence, we think it worthwhile to examine the kinetics of dissolution and leaching of salts from soils in regions where the water table is deep, i.e., where there is no drainage and the vertical salt distribution is nonuniform [2] (Fig. 1).

In the simplest case the basic equation of motion of the salts will be [3, 4]:

$$\frac{\partial C}{\partial t} = D^* \frac{\partial^2 C}{\partial x^2} - V \frac{\partial C}{\partial x} + \beta (C_s - C) \quad \left(V_0 = \frac{V}{m}\right). \tag{1}$$

Here C is the concentration of the soil solution (g/l), t is the time (days), x is the distance (m), V is the actual velocity of water in the pores of the ground, V_0 is the filtration velocity (m/day), m is the porosity, C_s is the limiting saturation concentration (g/l), β is the dissolution coefficient (1/day), and D^* is the coefficient of convective (filtration) diffusion (m^2/day).

This equation assumes linear (unidimensional) motion of the salts and water along the x-axis, a constant filtration velocity V_0 = const, and that the rate of dissolution of salts contained in the solid phase of the soil is independent of their volume and surface. If these conditions are not fulfilled Eq. (1) becomes more complex [4].

We will seek the solution of Eq. (1) with the following initial and boundary conditions:

when $t = 0$

$$C = C_0 + (C_1 - C_0)\, x / h_1 \qquad (0 \leqslant x \leqslant h_1),$$

$$C = C_2 + (C_1 - C_2)(h_2 - x) / (h_2 - h_1) \quad (h_1 \leqslant x \leqslant h_2),$$

$$C = C_2 \quad (h_2 \leqslant x < \infty)$$

when $t > 0$

$$V (C - C_n) = D^* \frac{\partial C}{\partial x} \ (x = 0),$$

$$\frac{\partial C}{\partial x} = 0 \ (C = \text{const}) \ (x \to \infty). \tag{2}$$

Here j are characteristic points of initial salinity (Fig. 1), C_j is the concentration at these points (g/l), h_j is the distance from the soil surface to the point j (m), and C_i is the salt concentration in the irrigation water (g/l).

The substitution [5]

$$C(x, t) = C_s + u(x, t) \exp\left[\frac{Vx}{2D^*} - \left(\frac{V^2}{4D^*} + \beta\right) t\right] \tag{3}$$

brings Eq. (1) and conditions (2) to the form

$$\partial u / \partial t = D^* \partial^2 u / \partial x^2, \tag{4}$$

when $t = 0$

$$u = \left(C_0 - C_s + \frac{(C_1 - C_0)\, x}{h_1}\right) \exp\left(-\frac{Vx}{2D^*}\right) \quad (0 \leqslant x \leqslant h_1),$$

$$u = \left(C_2 - C_s + \frac{(C_1 - C_2)(h_2 - x)}{h_2 - h_1}\right) \exp\left(-\frac{Vx}{2D^*}\right)$$

$$(h_1 \leqslant x \leqslant h_2),$$

$$u = (C_2 - C_s) \exp\left(-\frac{Vx}{2D^*}\right) \quad (h_2 \leqslant x < \infty),$$

when $t > 0$

$$D^* \frac{\partial u}{\partial x} - {}^1\!/_2\, V u = V (C_s - C_i) \exp\left(\frac{V^2}{4D^*} + \beta\right) t \quad (x = 0),$$

$$D^* \frac{\partial u}{\partial x} + {}^1\!/_2\, V u = 0 \quad (x \to \infty). \tag{5}$$

Instead of the function u(x, t) we introdude a new function by means of the equality

$$u(x, t) = w(x, t) + \frac{1}{2\sqrt{\pi t D^*}} \int_0^\infty f(\zeta) \exp\left(-\frac{(x - \zeta)^2}{4tD^*}\right) d\zeta. \tag{6}$$

Here $f(\zeta)$ is a function specifying the initial salt distribution, and then

$$u(x, t) = w(x, t) +$$

$$+ \frac{1}{2\sqrt{\pi t D^*}} \left\{ \int_0^{h_1} \left(C_0 - C_s + \frac{C_1 - C_0}{h_1} \zeta\right) f_0(\zeta)\, d\zeta + \right.$$

$$+ \int_{h_1}^{h_2} \left[C_2 - C_s + \frac{C_1 - C_2}{h_2 - h_1}(h_2 - \zeta)\right] f_0(\zeta)\, d\zeta +$$

$$\left. + (C_2 - C_s) \int_{h_2}^{\infty} f_0(\zeta)\, d\zeta \right\}$$

$$f_0(\zeta) = \exp(-V\zeta / 2D^* - (x - \zeta)^2 / 2tD^*). \tag{7}$$

Omitting the transformation calculations for u(x, t), we obtain

$$u(x, t) = w(x, t) + \frac{1}{2} \exp\left(\frac{V^2 t}{4D^*} - \frac{Vx}{2D^*}\right) \times$$

$$\times \left\{ \left[C_0 - C_s + \frac{C_1 - C_0}{h_1}(x - Vt)\right] f_1(0; x) + \right.$$

$$+ \left[C_2 - C_1 + \frac{C_1 - C_0}{h_1}(Vt - x) + \right.$$

$$\left. + \frac{C_1 - C_2}{h_2 - h_1}(h_2 - x + Vt)\right] f_1(h_1; x) -$$

$$- \frac{C_1 - C_2}{h_2 - h_1}(h_2 - x + Vt)\, f_1(h_2; x) +$$

$$+ 2\left(\frac{tD^*}{\pi}\right)^{1/2} \left[\frac{C_1 - C_0}{h_1} f_2(0; x) - \right.$$

$$\left. - \left(\frac{C_1 - C_0}{h_1} + \frac{C_1 - C_2}{h_2 - h_1}\right) f_2(h_1; x) + \frac{C_1 - C_2}{h_2 - h_1} f_0(h_2; x)\right],$$

$$f_1(h_2; x) = \operatorname{erfc}({}^1\!/_2\, V \sqrt{t/D^*} +$$

$$+ (h_2 - x) / 2\sqrt{tD^*}); \ f_2(h_2; x) =$$

$$= \exp[-({}^1\!/_2\, V \sqrt{t/D^*} + (h_2 - x) / 2\sqrt{tD^*})^2]. \tag{8}$$

Then Eq. (4) and condition (5) take the form

$$\partial w / \partial t = D^* \partial^2 w / \partial x^2 \ (0 \leqslant x < \infty), \ w = 0 \ (t = 0), \tag{9}$$

when $t > 0$, $x = 0$

$$D^* \frac{\partial w}{\partial x} - \frac{V}{2} w =$$

$$V(C_s - C_i)\exp\left[\left(\frac{V^2}{4D^*}+\beta\right)t\right] + {}^1/_2 \exp\frac{V^2 t}{4D^*}\times$$

$$\times\left\{\left[V(C_0 - C_s) + (V^2 t + D^*)\frac{C_1 - C_0}{h_1}\right] f_1(0; 0) + \right.$$

$$+\left[V(C_2 - C_0) + (V^2 t + D^*)\frac{C_1 - C_0}{h_1} + \right.$$

$$\left. + (Vh_2 + V^2 t + D^*)\frac{C_1 - C_2}{h_2 - h_1}\right] f(h_1; 0) -$$

$$-(Vh_2 + V^2 t + D^*)\frac{C_1 - C_2}{h_2 - h_1} f_1(h_2; 0) + \left(\frac{D^*}{\pi t}\right)^{1/2}(C_s - C_0) +$$

$$+2V\left(\frac{tD^*}{\pi}\right)^{1/2}\frac{C_1 - C_2}{h_1}\Big] f_2(0; 0) - 2V\left(\frac{tD^*}{\pi}\right)^{1/2}\left[\frac{C_1 - C_0}{h_1} + \right.$$

$$\left.\left. + \frac{C_1 - C_2}{h_2 - h_1}\right] f_2(h_1; 0) + 2V\left(\frac{tD^*}{\pi}\right)^{1/2}\frac{C_1 - C_2}{h_2 - h_1} f_2(h_2; 0)\right\}, \qquad (10)$$

when $t > 0$, $x \to \infty$

$$D^*\partial w / \partial x + {}^1/_2 V w = 0.$$

Henceforth, we denote the right-hand side of Eq. (10.1) by $f(t)$.

We apply the Laplace-Carson transformation [6, 7] to Eqs. (9) and (10)

$$F(x, p) = p\int_0^\infty f(x, \tau)\exp(-p\tau)\, d\tau. \qquad (11)$$

After transformation we obtain, writing in the same order,

$$D^* d^2W / dx^2 - pW = 0 \quad (0 \leqslant x < \infty),$$

$$p\frac{V}{2}W - pD^*\frac{dW}{dx} = F(p) \quad (x = 0),$$

$$\frac{V}{2}W + D^*\frac{dW}{dx} = 0 \quad (x \to \infty). \qquad (12)$$

The solution of the first equation of (12) is [8]

$$W = A\exp\left(x\sqrt{p / D^*}\right) + B\exp\left(-x\sqrt{p / D^*}\right). \qquad (13)$$

From the second and third equations of (12) we have

$$A = 0,\ B = -\frac{F(p)}{p\sqrt{D^*}H(p)} \quad \left(H(p) = \frac{V}{2\sqrt{D^*}} + \sqrt{p}\right). \qquad (14)$$

For W(x,p), omitting simple transformations, we find

$$W(x, p) = -\frac{F(p)F_1(x; 0)}{p\sqrt{D^*}H(p)}\times$$

$$\times\left(F_1(x; h_2) = \exp\left[-\frac{Vh_2}{2D^*} - (h_2 + x)\left(\frac{p}{D^*}\right)^{1/2}\right]\right). \qquad (15)$$

Taking into account that

$$F(p) = \frac{Vp(C_s - C_i)}{p - (V^2/4D^* + \beta)} +$$

$$+\frac{\sqrt{p}(C_s - C_0)}{2}\left(\sqrt{D^*} - \frac{V}{H(p)}\right) + \frac{\sqrt{pD^*}(C_1 - C_0)}{2h_1 H(p)}\times$$

$$\times\left[\frac{V(1 - F_1(0; h_1))}{H(p)} + \sqrt{D^*}(F_1(0; h_1) - 1)\right] +$$

$$+\frac{\sqrt{pD^*}(C_1 - C_2)}{2(h_2 - h_1)H(p)}\left[\frac{V(F_1(0; h_2) - F_1(0; h_1))}{H(p)} + \right.$$

$$\left. + \sqrt{D^*}(F_1(0; h_2) - F_1(0; h_2))\right], \qquad (16)$$

we obtain the solution in images

$$W(x, p) = \frac{V(C_{\text{п}} - C_{\text{н}})F_1(x; 0)}{\sqrt{D^*}\left[p - (V^2/2D^* + \beta)H(p)\right]} +$$

$$+\frac{\sqrt{p}(C_0 - C_{\text{н}})F_1(x; 0)}{2H(p)}\left[1 - \frac{V}{\sqrt{D^*}H(p)}\right] +$$

$$+\frac{\sqrt{p}(C_0 - C_1)}{2h_1 H^2(p)}\left[\frac{V(F_1(x; 0) - F_1(x; h_1))}{H(p)} + \right.$$

$$\left. + \sqrt{D^*}(F_1(x; h_1) - F_1(x; 0))\right] +$$

$$+\frac{\sqrt{p}(C_2 - C_1)}{2(h_2 - h_1)H^2(p)}\left[\frac{V(F_1(x; h_2) - F_1(x; h_1))}{H(p)} + \right.$$

$$\left. + \sqrt{D^*}(F_1(x; h_1) - F_1(x; h_2))\right]. \qquad (17)$$

Converting to the pre-image and considering Eqs. (8) and (3) consecutively, we obtain the solution of the problem (1) and (2)

$$C = C_s + (C_i - C_s)\left[\frac{\omega_-(z)}{1 + b} + \frac{\omega_+(z)}{1 - b} - \frac{2\chi(z)}{1 - b^2}\right] +$$

$$+\exp\left[-a(b^2 - 1)\right]\frac{C_0 - C_{\text{н}}}{2}\operatorname{erfc} az_2 +$$

$$+4a^2 z_1\exp(4a^2 z)\operatorname{erfc} az_1 - \frac{4a}{\sqrt{\pi}}\exp(-a^2 z_2^2) +$$

$$+\frac{C_0 - C_1}{2(z_3 - z_1)}\left[\varphi(z_1) - \varphi(z_3) + z_2(\operatorname{erfc} az_2 - \operatorname{erfc} az_4) + \right.$$

$$\left. + \frac{1}{a\sqrt{\pi}}\left(\psi(z_3) - \psi(z_1) + \exp(-a^2 z_4^2) - \exp(-a^2 z_2^2)\right)\right] +$$

$$+\frac{C_2 - C_1}{2(z_5 - z_3)}\left[\varphi(z_5) - \varphi(z_3) - z_6(\operatorname{erfc} az_4 - \operatorname{erfc} az_6) + \right.$$

$$\left. + \frac{1}{a\sqrt{\pi}}\left(\psi(z_3) - \psi(z_5) + \exp(-a^2 z_4^2) - \exp(-a^2 z_6^2)\right)\right] +$$

$$+\frac{C_2 - C_0}{2}\operatorname{erfc} az_4,$$

$$\omega_\pm(z) = \exp\left[\pm 2a^2 z(1 + b)\right]\operatorname{erfc} a(1 \pm b),$$

$$\varphi(z_k) = (2a^2 z_k^2 + z_k + 1)\exp(4a^2 z)\operatorname{erfc} az_k,$$

$$\psi(z_k) = (2a^2 z_k + 1)\exp\left[-a^2(z_k^2 - 4z)\right] \quad (k = 1.3.5),$$

$$a = {}^1/_2 V\sqrt{tD^*},\ \chi(z) = \exp\left[a^2(1 + \right.$$

$$\left. + 4z - b^2)\right]\operatorname{erfc} a(1 + z),$$

$$z = x^\circ \quad (x^\circ = x / Vt),$$

$$z_1 = 1 + x^\circ,\ z_3 = 1 + h_1^\circ + x^\circ \ (h_1^\circ = h_1 / Vt),$$

$$z_4 = 1 + h_1^\circ - x^\circ,\ z_5 = 1 + h_2^\circ + x^\circ \ (h_2^\circ = h_2 / Vt),$$

$$z_6 = 1 + h_2^\circ - x^\circ,\ b = \sqrt{1 + \Pi},\ z_2 = 1 - x^\circ. \qquad (18)$$

Here $\Pi = 4\beta D^*/V^2$ is a parameter characterizing the drift from the site of dissolution [9].

For readily soluble salts which are present in small amounts in the solid phase (chlorine salts, for instance) Eq. (1) satisfactorily predicts the natural and experimentally observed salt distribution without the last term, i.e.,

$$\partial C / \partial t = D^*\partial^2 C / \partial x^2 - V\partial C / \partial x. \qquad (19)$$

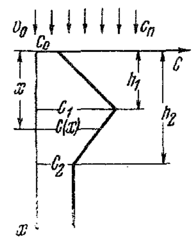

Fig. 1

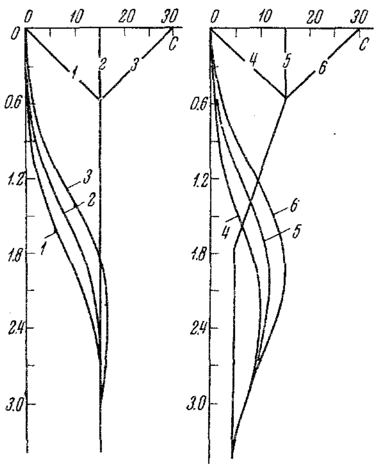

Fig. 2

Then the solution of Eq. (19) with boundary and initial conditions (2) has the form

$$C = C_i + \frac{C_0 - C_i}{2}\Big[\operatorname{erfc} az_2 + (4a^2z_1 + 1)\exp(4a^2z)\operatorname{erfc} az_1 - \frac{4a}{\sqrt{\pi}}\exp(-a^2z_2^2)\Big] + \frac{C_0 - C_1}{2(z_3 - z_1)}\Big\{\varphi(z_1) - \varphi(z_3) + z_2(\operatorname{erfc} az_2 - \operatorname{erfc} az_4) + \frac{1}{a\sqrt{\pi}}[\psi(z_3) - \psi(z_1) + \exp(-a^2z_4^2) - \exp(-a^2z_2^2)]\Big\} + \frac{C_2 - C_0}{2(z_5 - z_3)}\Big\{\varphi(z_5) - \varphi(z_3) - z_6(\operatorname{erfc} az_4 - \operatorname{erfc} az_6) + \frac{1}{a\sqrt{\pi}}[\psi(z_3) - \psi(z_5) + \exp(-a^2z_4^2) - \exp(-a^2z_6^2)]\Big\} + \frac{C_2 - C_0}{2}\operatorname{erfc} az_4. \tag{20}$$

If the initial salt distribution is more complex than in Fig. 1, the solution has the form

$$C = C_i + \frac{C_0 - C_i}{2}[\operatorname{erfc} az_2 - \exp(4a^2z)(\operatorname{erfc} az_1 - 4a\, i\operatorname{erfc} az_1)] + \sum_{j=0}^{n}\frac{C_j - C_{j+1}}{2a(h_{j+1}^\circ - h_j^\circ)}[i\operatorname{erfc} az_{j+1}^- - i\operatorname{erfc} az_j^- + \exp(4a^2z)(i\operatorname{erfc} az_{j+1}^+ - i\operatorname{erfc} az_j^+) + 4a\exp(4a^2z)(i^2\operatorname{erfc} az_j^+ - i^2\operatorname{erfc} z_{j+1}^+)],$$

$$z_j^+ = 1 + h_j^\circ + x^\circ,\quad z_j^- = 1 + h_j^\circ - x^\circ. \tag{21}$$

Thus, the solution (18), (20), and (21) can be used to predict the redistribution of salts for several different initial salt distributions.

If salinity is uniform before irrigation, the solution has the form ([1], and others)

$$C = C_i + \frac{C_0 - C_i}{2}\Big[\operatorname{erfc} az_2 + (4a^2z_1 + 1)\times \exp(4a^2z)\operatorname{erfc} az_1 - \frac{4a}{\sqrt{\pi}}\exp(-a^2z_2^2)\Big].$$

To illustrate the solution for several characteristic initial salinity distributions, calculations were completed with the following data: $D^* = 2.34\cdot10^{-3}$ m^2/day, $V_0 = 0.01$ m/day, m = 0.4, $C_i = 0$. The graphs were drawn for t = 60 days.

The graphs in Fig. 2 clearly illustrate the redistribution of salts after irrigation in relation to the initial salinity distribution.

The author thinks that the proposed formulas can be used to predict the redistribution of salts in soil with due regard to their properties and the properties of the soils.

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10 January 1967 Moscow